

Are nonlinear discrete cellular automata compatible with quantum mechanics?

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Abstract. We consider discrete and integer-valued cellular automata (CA). A particular class of which comprises “Hamiltonian CA” with equations of motion that bear similarities to Hamilton’s equations, while they present discrete updating rules. The dynamics is *linear*, quite similar to unitary evolution described by the Schrödinger equation. This has been essential in our construction of an invertible *map* between such CA and continuous quantum mechanical models, which incorporate a fundamental *discreteness* scale. Based on Shannon’s sampling theory, it leads, for example, to a one-to-one relation between quantum mechanical and CA conservation laws. The important issue of linearity of the theory is examined here by incorporating higher-order nonlinearities into the underlying action. These produce inconsistent nonlocal (in time) effects when trying to describe continuously such nonlinear CA. Therefore, in the present framework, only linear CA and local quantum mechanical dynamics are compatible.

1. Introduction

A novel analysis of quantum mechanics (QM) — which aims at redesigning the foundations of quantum theory — has recently been proposed by G. ’t Hooft [1]. The hope for a comprehensive theory expressed in this far-reaching considerations is founded on the observation that quantum mechanical features arise in a large variety of deterministic “mechanical” models. While practically all of these models have been singular cases, i.e., which cannot easily be generalized to cover a realistic range of phenomena incorporating interactions, CA promise to provide the necessary versatility [2, 3, 4]. For an incomplete list of various earlier attempts in this field, see, for example, Refs. [5, 6, 7, 8, 9, 10, 11, 12, 13] and further references therein.

The linearity of quantum mechanics (QM) is a fundamental feature, which is particularly visible in the Schrödinger equation (leaving aside here models which attempt to describe measurement processes dynamically). This is independent of the particular object under study, provided it is sufficiently isolated from anything else, and is naturally reflected in the superposition principle. Thus, linearity entails the “quantum essentials” interference and entanglement.

The linearity of QM has been questioned and nonlinear modifications have been proposed earlier — not only as suitable approximations for complicated many-body dynamics, but especially in order to test experimentally the robustness of QM against such *nonlinear deformations*. This has been thoroughly discussed by T.F. Jordan who presented a proof ‘from within’ quantum theory that the theory has to be linear, given the *separability* assumption

“... that the system we are considering can be described as part of a larger system without interaction with the rest of the larger system.”[14]

In our recent work, we have considered a *discrete dynamical theory*, which deviates notably from quantum theory, at first sight. However, we have shown with the help of sampling theory that the deterministic mechanics of the class of Hamiltonian CA can be related to QM in the presence of a fundamental time scale. This relation appears to demonstrate that consistency of the action principle of the underlying discrete dynamics implies, in particular, the linearity of both theories.

We will review some pertinent results, followed by a more detailed study of the undesirable consequences that we are confronted with, if we generalize the action principle by incorporating *genuine nonlinearities* and try to enlarge the proposed class of CA in this way.

The CA approach may offer additional insight into interference and entanglement, in the limit where the discreteness scale can be treated as sufficiently small. Furthermore, future developments are conceivable which can address the *dynamics* of QM measurement processes, about which standard quantum theory remains silent.

2. An action principle for cellular automata

In the following, we recall briefly the dynamics of a class of discrete CA automata introduced earlier, which bears remarkable similarity with Hamiltonian dynamics in the continuum on one side and with QM on the other [2, 3, 4].

We describe the state of a classical CA with countably many degrees of freedom by discrete *integer-valued* “coordinates” x_n^α, τ_n and “conjugated momenta” p_n^α, π_n , where $\alpha \in \mathbf{N}_0$ denote different degrees of freedom and $n \in \mathbf{Z}$ different states. The x_n and p_n might be higher-dimensional vectors, while τ_n and \mathcal{P}_n are assumed one-dimensional. — The “coordinate” τ_n has been separated from the x_n^α ’s (correspondingly π_n from the p_n^α ’s), since this degree of freedom allows to represent a *dynamical time* variable, discussed in [2, 3, 15, 16], with further references given there.

For any one of the dynamical variables, say f_n , the finite difference operator Δ is defined by:

$$\Delta f_n := f_n - f_{n-1} . \quad (1)$$

Furthermore, we introduce the quantities (assuming the summation convention for Greek indices, $r^\alpha s^\alpha \equiv \sum_\alpha r^\alpha s^\alpha$):

$$a_n := c_n \pi_n , \quad (2)$$

$$H_n := \frac{1}{2} S_{\alpha\beta} (p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta} p_n^\alpha x_n^\beta + R_n , \quad (3)$$

$$\mathcal{A}_n := \Delta \tau_n (H_n + H_{n-1}) + a_n , \quad (4)$$

where the constants, c_n , the symmetric, $\hat{S} \equiv \{S_{\alpha\beta}\}$, and the antisymmetric, $\hat{A} \equiv \{A_{\alpha\beta}\}$, matrices are all integer-valued. The definition (2) determines the behaviour of the variable τ_n and only the most simple choice (involving a single constant) will be relevant for our purposes, cf. below. R_n stands for higher than second powers in x_n^α or p_n^α . — Discussion of such *genuine nonlinearities* is the aim of the present note and will be resumed in due course.

Based on these definitions, we introduce the *integer-valued CA action* by:

$$\mathcal{S} := \sum_n [(p_n^\alpha + p_{n-1}^\alpha) \Delta x_n^\alpha + (\pi_n + \pi_{n-1}) \Delta \tau_n - \mathcal{A}_n] . \quad (5)$$

Furthermore, we consider *integer-valued variations* δf_n to be applied to a polynomial g :

$$\delta_{f_n} g(f_n) := [g(f_n + \delta f_n) - g(f_n - \delta f_n)] / 2\delta f_n , \quad (6)$$

and $\delta f_n g \equiv 0$, if $\delta f_n = 0$. — In terms of these notions, we postulate the following variational principle for the CA dynamics.

The CA Action Principle: The discrete evolution of a CA is determined by the stationarity of its action under arbitrary integer-valued variations of all dynamical variables, $\delta \mathcal{S} = 0$. •

Some characteristic features of this *CA Action Principle* deserve to be mentioned:

- i) Variations of terms that are *constant, linear, or quadratic in integer-valued variables* yield analogous results as standard infinitesimal variations of corresponding terms in the continuum.
- ii) While infinitesimal variations do not conform with integer valuedness, there is *a priori* no restriction of integer variations, hence *arbitrary integer-valued variations* must be admitted.
- iii) However, for arbitrary variations δf_n , the *remainder of higher powers*, R_n in H_n , which enters the action, has to vanish for consistency. Otherwise the number of equations of motion generated by variation of the action, according to Eq.(6), would exceed the number of variables. (A suitably chosen R_0 or a sufficient small number of such remainder terms can serve to encode the *initial conditions* for CA dynamics.)

As we observed in earlier work, these features seem to be essential when constructing a map between the considered CA and equivalent quantum mechanical models based on sampling theory, *cf.* Sec.3. We will try to illuminate this in Sec.4., by studying a generalization of the variational principle incorporating nonlinearities and by pointing out consequences of such nonlinearities which obstruct this way to arrive at QM models from CA.

2.1. The CA equations of motion and conservation laws

We now apply the *CA Action Principle* to the action \mathcal{S} (keeping $R_n \equiv 0$ for the moment), which yields the following discrete equations of motion:

$$\dot{x}_n^\alpha = \dot{\tau}_n (S_{\alpha\beta} p_n^\beta + A_{\alpha\beta} x_n^\beta) , \quad \dot{p}_n^\alpha = -\dot{\tau}_n (S_{\alpha\beta} x_n^\beta - A_{\alpha\beta} p_n^\beta) , \quad (7)$$

$$\dot{\tau}_n = c_n , \quad \dot{\pi}_n = \dot{H}_n , \quad (8)$$

introducing the suggestive notation $\dot{O}_n := O_{n+1} - O_{n-1}$. We emphasize that all terms herein are integer-valued. The fact that we arrive at *finite difference equations* reflects the discreteness of the *automaton time* n and their appearance has motivated the name *Hamiltonian CA*.

The equations of motion are *time reversal invariant*, since the state $n+1$ can be calculated from knowledge of the earlier states n and $n-1$ and the state $n-1$ from the later ones $n+1$ and n . — Note that the $\dot{\tau}_n$ present parameters for the evolving x, p -variables, as a consequence of Eqs. (8). More generally, $\dot{\tau}$ can play the role of a dynamically coupled *lapse function* in Eqs. (7).

Introducing the self-adjoint matrix $\hat{H} := \hat{S} + i\hat{A}$, the Eqs. (7) can be combined into:

$$\dot{x}_n^\alpha + i\dot{p}_n^\alpha = -i\dot{\tau}_n H_{\alpha\beta} (x_n^\beta + ip_n^\beta) , \quad (9)$$

and its adjoint. Thus, we find here a *discrete analogue of Schrödinger's equation*, with $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$ as amplitude of the “ α -component” of “state vector” $|\psi\rangle$ at “time” n and with \hat{H} as *Hamilton operator*. (We will use QM terminology freely here and in the following.)

Furthermore, there are *conservation laws* respected by the discrete equations of motion, or by Eq. (9), which are in *one-to-one correspondence* with those of the corresponding Schrödinger equation in the continuum [2, 3, 4]. — In particular, the Eqs. (7) imply the following theorem.

Theorem A: For any matrix \hat{G} that commutes with \hat{H} , $[\hat{G}, \hat{H}] = 0$, there is a *discrete conservation law*:

$$\psi_n^{*\alpha} G_{\alpha\beta} \dot{\psi}_n^\beta + \dot{\psi}_n^{*\alpha} G_{\alpha\beta} \psi_n^\beta = 0 . \quad (10)$$

For self-adjoint \hat{G} , with complex integer elements, this relation concerns real integers. •

Corollary A: For $\hat{G} := \hat{1}$, the Eq. (10) implies a *conserved constraint* on the state variables:

$$\psi_n^{*\alpha} \dot{\psi}_n^\alpha + \dot{\psi}_n^{*\alpha} \psi_n^\alpha = 0 . \quad (11)$$

For $\hat{G} := \hat{H}$, an *energy conservation* law follows. •

Note that Eqs.(10) and (11) *cannot* be “integrated” as usual, since the *Leibniz rule* is modified here. Recalling $\dot{O}_n := O_{n+1} - O_{n-1}$, we find, for example, $O_{n+1}O'_{n+1} - O_{n-1}O'_{n-1} = \frac{1}{2}(\dot{O}_n[O'_{n+1} + O'_{n-1}] + [O_{n+1} + O_{n-1}]\dot{O}'_n)$, instead of the product rule of differentiation.

Furthermore, the continuum limit of the equations of motion and their conservation laws does not simply follow from letting the discreteness scale $l \rightarrow 0$. The integer valuedness of all quantities conflicts with continuous time and related derivatives. Hence, we need a more elaborate mapping, in order to relate CA to continuum models.

In the following Sec. 3., it will be shown that such an *invertible map* between the descriptions of discrete time Hamiltonian CA and of quantum mechanical objects evolving in continuous time can indeed be constructed, taking into account a fundamental discreteness scale l .

3. The CA \leftrightarrow QM map based on sampling theory

Despite the notable similarities of the Hamiltonian CA with QM systems, we may wonder whether the discreteness of a CA can be reconciled with a continuum description at all and, in particular, with QM?

We have suggested earlier that especially wave functions, like other fields, could be *simultaneously discrete and continuous*, represented by sufficiently smooth functions containing a finite density of degrees of freedom [2, 3]. Related ideas have been discussed by T.D. Lee and collaborators and recently by A. Kempf in attempts to introduce a covariant ultraviolet cut-off into quantum field theories and, last not least, for gravity, see, for example, Refs. [15, 17] with further references there (and in [2, 3]). However, *integer-valued CA* have first been connected with *structure of QM* from this perspective in our work.

Information can have simultaneously continuous and discrete character as pointed out by C.E. Shannon in his pioneering work [18]. This is routinely applied in signal processing, converting analog to digital encoding and *vice versa*. Sampling theory demonstrates that a bandlimited signal can be perfectly reconstructed, provided discrete samples of it are taken at the rate of at least twice the band limit (Nyquist rate) – for an extensive review, see Ref. [19].

We shall make use of the basic version [19] of the Sampling Theorem: Consider square integrable *bandlimited functions* f , *i.e.*, which can be represented as $f(t) = (2\pi)^{-1} \int_{-\omega_{max}}^{\omega_{max}} d\omega e^{-i\omega t} \tilde{f}(\omega)$, with bandwidth ω_{max} . Given the set of amplitudes $\{f(t_n)\}$ for the set $\{t_n\}$ of equidistantly spaced times (spacing π/ω_{max}), the function f is obtained for all t by:

$$f(t) = \sum_n f(t_n) \frac{\sin[\omega_{max}(t - t_n)]}{\omega_{max}(t - t_n)} . \quad (12)$$

Since the CA state is labelled by the integer n , the *automaton time*, the corresponding discrete *physical time* is obtained by multiplying with the fundamental scale l , $t_n \equiv nl$, and the bandwidth by $\omega_{max} = \pi/l$.

When attempting to *map invertibly* Eqs. (7) on continuum equations, according to Eq. (12), the nonlinearity on the right-hand sides is problematic, since the product of two functions, with bandwidth ω_{max} each, is not a function with the same bandwidth. Therefore, we assume here that $\dot{\tau}_n$ is a constant and postpone the discussion of generic nonlinearities to Sec. 4.

Recalling the discrete time equation (9), we insert $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$, as before, and apply the *Sampling Theorem* to obtain the equivalent *continuous time equation*:

$$(\hat{D}_l - \hat{D}_{-l})\psi^\alpha(t) = 2 \sinh(l\partial_t)\psi^\alpha(t) = \frac{1}{i}H_{\alpha\beta}\psi^\beta(t) , \quad (13)$$

employing the translation operator defined by $\hat{D}_T f(t) := f(t+T)$ and implementing the natural choice $\dot{\tau}_n \equiv 1$, for all n .

It appears that we recover the *Schrödinger equation*. However, it is modified in important ways, reflecting the presence of the scale l .

First of all, by construction, the continuous time wave function ψ^α is bandlimited (by ω_{max}). Therefore, knowing the wave function at the discrete times of a set $\{t_0 + nl | n \in \mathbf{Z}\}$, with t_0 arbitrary, it can be reconstructed for all times by a slight generalization of Eq. (12). Furthermore, since Eq.(13) is of the form $f(t+l) = f(t-l) - i\hat{H}f(t)$, it is sufficient to know f at two times, say t_0 and $t_0 - l$, in order to obtain it for all times of the set $\{t_0 + nl | n \in \mathbf{Z}\}$. Thus, we learn that *two initial conditions* (separated by a time step l) have to be specified to define the solution of Eq. (13). This agrees precisely with the requirements of the discrete description of the CA, cf. Sec. 2.1. — In order that sampling reproduces the *integer-valuedness* of the CA, both initial values have to be integer-valued.

If instead the modified Schrödinger equation is written in terms of the infinite series of odd powers of time derivatives, it might give the false impression that an infinity of initial conditions are required. In any case, the higher-order derivatives are negligible for low-energy wave functions, which vary little with respect to the cut-off scale, *i.e.* $|\partial^k \psi / \partial t^k| \ll l^{-k} = (\omega_{max}/\pi)^k$.

Secondly, the bandlimit $|\omega| \leq \omega_{max}$ leads to an *ultraviolet cut-off* of the energy E of stationary states of the generic form $\psi_E(t) := \exp(-iEt)\tilde{\psi}$. Diagonalizing the self-adjoint Hamiltonian, yields an eigenvalue equation and a *modified dispersion relation*:

$$E_\alpha = l^{-1} \arcsin(\epsilon_\alpha/2) = (2l)^{-1} \epsilon_\alpha [1 + \epsilon_\alpha^2/24 + O(\epsilon_\alpha^4)] , \quad (14)$$

where $\{\epsilon_\alpha\}$ is the set of eigenvalues of the Hamiltonian; *e.g.*, α could label the momentum modes of a given spatial lattice. We find that the spectrum $\{E_\alpha\}$ is cut off by $|E_\alpha| \leq \pi/2l = \omega_{max}/2$, *i.e.* half the bandlimit. We have discussed further aspects of this result elsewhere [4, 20].

Finally, the discrete CA conservation laws, *Theorem A* and *Corollary A* in Sec. 2.1., Eqs. (10) and (11), respectively, naturally have a counterpart in the continuum description obtained with the help of sampling theory. The absence of an ordinary time derivative in the discrete CA model, where only finite differences can play a role, leads to similar combinations of translation operators as on the left-hand side of the modified Schrödinger equation, Eq. (13), in the continuous time conservation laws. See Refs. [3, 4], where also related symmetries have been addressed.

4. The options for nonlinear Hamiltonian cellular automata

The general properties and certain quantum features, in particular, of the Hamiltonian CA that we recalled in the previous sections derive from the *CA Action Principle* introduced in Sec. 2. A most important aspect that we observed has been the *linearity* of the equations of motion, Eqs. (7) or Eq. (9).

We have repeatedly pointed out that additional higher-order terms in the action, which would lead to nonlinear (in x_n^α, p_n^α or ψ_n^α) terms in the equations of motion, would simultaneously enlarge the set of equations. This differs markedly from what one is used to in applications of variational principles based on the continuum, where ordinary differential calculus is available. Presently, due to the absence of infinitesimals and ensuing necessity to admit arbitrary variations, the CA dynamics tends to become overdetermined. — As we shall see in the following, however, a closer look at this problem reveals some interesting information on the availability and consequences of nonlinear extensions.

4.1. Generalizing the variational derivative

The additional equations of motion, which can render the equations of motion inconsistent since overdetermined, are related to two aspects of the dynamics. One comprises the additional higher than quadratic powers of dynamical variables in the action, which kind of terms we summarized by R_n in definition (3). While the other consists in the *arbitrary integer-valued variations* δf_n of all dynamical variables present in the action, which are allowed by the variational principle. They are applied according to the definition of Eq. (6) and this can produce additional terms which involve powers of δf_n . The coefficients of such terms all have to vanish independently, leading to additional equations of motion. The resulting enlarged set of equations, as compared to a given number of variables, will be overdetermined in generic cases.

This problem can be avoided by suitably generalizing the definition of the variations. Replacing Eq. (6), we define here:

$$\delta_f g^{(N)}(f) := \sum_{k \geq 1} \gamma_k [g^{(N)}(f + m_k \delta f) - g^{(N)}(f - m_k \delta f)] / 2 \delta f \quad , \quad (15)$$

where f stands for a dynamical variable entering the N -th degree polynomial $g^{(N)}$ and γ_k and m_k ($m_k \neq m_{k'}$, for $k \neq k'$) are constant real and positive integer-valued coefficients, respectively, to be determined. — Namely, we aim to arrange these coefficients in such a way that $\delta_f g^{(N)}(f) = \bar{g}^{(N-1)}(f)$; thus, the variation results in a polynomial $\bar{g}^{(N-1)}$ of degree $N - 1$ and all other possible terms proportional to powers of δf *cancel by construction*.

This eliminates the possibility of having an overdetermined set of (generally nonlinear) equations of motion. — In order to proceed, we write the polynomial $g^{(N)}$ explicitly,

$$g^{(N)}(f) := g_0 + g_1 f^1 + \dots + g_N f^N \quad , \quad (16)$$

and expand the difference appearing in Eq. (15):

$$\begin{aligned} [g^{(N)}(f + m_k \delta f) - g^{(N)}(f - m_k \delta f)] / 2 &= g_1 \cdot m_k \delta f + g_2 \cdot 2m_k f \delta f \\ &+ g_3 \cdot (3m_k f^2 \delta f + (m_k \delta f)^3) \\ &+ g_4 \cdot (4m_k f^3 \delta f + 4f(m_k \delta f)^3) \\ &+ g_5 \cdot (5m_k f^4 \delta f + 10f^2(m_k \delta f)^3 + (m_k \delta f)^5) \\ &+ \dots \quad . \end{aligned} \quad (17)$$

Note that the terms $\propto \delta f$ correspond to the ones obtained by ordinary differentiation of the polynomial. However, since any one dynamical variable f of our CA is integer-valued, such derivatives are to be interpreted only as a symbolical notation in the following. Thus, we have $\delta_f g^{(N)}(f) = \sum_k \gamma_k m_k \cdot (d/df)g^{(N)}(f) + \dots$, where the additional terms involving powers of δf are not spelled out. The point of our considerations is that the latter can be made to vanish always by suitably adjusting the coefficients γ_k and m_k , if we restrict the maximal order of polynomials to be dealt with.

For illustration, we consider *all* polynomials of order ≤ 4 , *i.e.* $g^{(4)}$. Here, the terms $\propto \delta f^3$ are eliminated always, *cf.* Eq. (17), if the following condition is fulfilled:

$$\sum_{k \geq 1} \gamma_k (m_k)^3 \stackrel{!}{=} 0 \quad , \quad (18)$$

A solution is provided by: $m_1 = 1$, $m_2 = m \geq 2$, $\gamma_1 = 1/(1 - m^{-2})$, $\gamma_2 = -m^{-3}/(1 - m^{-2})$, and all other coefficients vanishing. Thus, we obtain: $\delta g^{(4)}(f) = (d/df)g^{(4)}(f)$, cf. Eq. (15). This solution is sufficient for our purposes but not unique.

By this elementary reasoning, we have obtained a satisfactory variational derivative, which avoids the problem of arriving at an overdetermined set of equations of motion. Our approach can be generalized to polynomials of arbitrary finite order; limitation to finite order being warranted by integer-valuedness of the variables.

Consequently, a suitably generalized variational derivative can be employed in the *CA Action Principle*, such that consistent finite difference *equations of motion incorporating nonlinear potentials* result, which maintain the structural similarity with classical Hamilton's equations that we have seen in Sec. 2.1.

4.2. Problems that arise with nonlinearity

The generalized variational derivative of Eq. (15) can be employed to define a *Poisson bracket*, similarly as we discussed elsewhere [4, 20] — Since this variational derivative acts practically like an ordinary derivative, there is apparently no need to restrict the related *algebra of observables* in any way.

Yet it does remain consistent to consider only (linear or) *quadratic forms* in the dynamical variables, recovering the previous results and analogous symmetry properties as in QM [20, 21].

However, once higher order polynomials (in x_n^α, p_n^α or ψ_n^α) are admitted in the action and equations of motion, or as observables, it is *not consistent* to limit the set of relevant polynomials at any finite order. For example, the Poisson bracket of two polynomials of order N and N' , respectively, may result in a polynomial of order $N + N' - 2 > N, N'$. This is problematic, since arbitrarily high powers of integer-valued variables, and linear combinations thereof, generated in this way, will eventually lead to *divergent quantities*!

Thus, we observe here a qualitatively profound ‘bifurcation’ in the properties of Hamiltonian CA tied to the presence or absence of nonlinearities in their equations of motion. The preceding remarks seem to imply that an algebra of observables cannot even be defined for the nonlinear case in an analogous way as in classical mechanics. To emphasize this point, we note that the kind of discrete or continuous *conservation laws* (and traces of QM unitary *symmetry*), discussed in Secs. 2.1. or 3., see also Ref. [3, 4], will generally be absent in nonlinear CA.

Further problematic features can be expected, when we consider the behaviour of nonlinear terms under the map relating the discrete description of CA and its continuum counterpart, employing Shannon's *Sampling Theorem* as in Sec. 3., to which we turn shortly.

4.2.1. A summary of properties of sinus cardinalis We will make use of several results concerning the *sinus cardinalis* function, $\text{sinc}(x) := \sin(x)/x$, which is employed in the reconstruction formula, Eq. (12), and will be needed in the following.

Let us define $s_n(t) := \text{sinc}[\pi(t/l - n)]$, which leads to the Fourier transform:

$$\int_{-\infty}^{\infty} dt e^{-i\omega t} s_n(t) = l\theta(\pi/l - \omega)\theta(\pi/l + \omega)e^{-i\omega ln} , \quad (19)$$

where θ denotes the Heaviside step function. Thus, the function s_n is *bandlimited*. Furthermore, it presents a “nascent” Dirac delta function, which has the properties:

$$l^{-1} \int_{-\infty}^{\infty} dt s_n(t) = 1 , \quad (20)$$

$$\lim_{l \rightarrow 0} l^{-1} \int_{-\infty}^{\infty} dt s_n(t)f(t) = f(0) . \quad (21)$$

These results can be recovered with the help of the Fourier transform of s_n , Eq. (19), assuming that f has a well-behaved Fourier transform. Employing the inverse Fourier transformation of Eq. (19), we obtain the orthogonality relation:

$$l^{-1} \int_{-\infty}^{\infty} dt s_m(t) s_n(t) = \delta_{mn} . \quad (22)$$

Finally, as an example, we perform a summation by applying the *Sampling Theorem*, Eq. (12):

$$\sum_{m \in \mathbf{Z}} s_n(ml - t') s_m(t) = s_n(t - t') , \quad (23)$$

noting that the factor of s_n under the sum can be interpreted as the function on the right-hand side sampled at the times ml ; all functions here have identical bandwidth, *i.e.* $\omega_{max} = \pi/l$.

4.2.2. Nonlinearity and nonlocality (in time) We now are ready to elaborate consequences of nonlinearity with respect to mapping the discrete equations of motion to the continuum description thereof *via* application of Shannon's *Sampling Theorem*. — Suppose that the discrete analogue of the Schrödinger equation, Eq. (9), includes a genuinely *nonlinear term*, *e.g.*:

$$\dot{\psi}_n^\alpha = -iH_{\alpha\beta} \psi_n^\beta + M_{\alpha\beta\gamma} (\psi_n^{*\beta} + \psi_n^\beta) (\psi_n^{*\gamma} + \psi_n^\gamma) , \quad (24)$$

keeping $\dot{\tau}_n \equiv 1$ and where the coefficients $M_{\alpha\beta\gamma}$ are real and totally symmetric in the indices.

A corresponding potential can be incorporated into the action and the additional nonlinearity in Eq. (24) follows by applying a suitably generalized variational derivative, as discussed in Sec. 4.1. We are not interested in the physical (ir)relevance of the present example of a cubic potential, but would like to see what happens with the nonlinear terms, *e.g.* $M_{\alpha\beta\gamma} \psi_n^\beta \psi_n^\gamma$, when the *Sampling Theorem* is applied to Eq. (24), similarly as before in Sec. 3.

Suppressing irrelevant greek indices, we introduce $\psi_n =: \psi(t_n)$ and $\psi_n \psi_n =: \psi_{(2)}(t_n)$. Through the reconstruction formula (12) the *discrete time* values $\psi(t_n)$ and $\psi_{(2)}(t_n)$ are replaced by *continuous time* functions $\psi(t)$ and $\psi_{(2)}(t)$, respectively, and we would like to make the relation between the latter functions explicit.

First of all, employing the orthogonality relation (22), we invert the reconstruction formula:

$$\psi(t_n) = l^{-1} \int_{-\infty}^{\infty} dt s_n(t) \psi(t) . \quad (25)$$

Which gives us simply:

$$\psi_{(2)}(t_n) = l^{-2} \int_{-\infty}^{\infty} dt' s_n(t') \psi(t') \int_{-\infty}^{\infty} dt'' s_n(t'') \psi(t'') . \quad (26)$$

Applying now the reconstruction formula to $\psi_{(2)}(t_n)$, we obtain indeed a nonlinear relation between $\psi_{(2)}(t)$ and $\psi(t)$:

$$\psi_{(2)}(t) = l^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt' dt'' \sum_{n \in \mathbf{Z}} s_n(t) s_n(t') s_n(t'') \psi(t') \psi(t'') , \quad (27)$$

where we interchanged summation and integrations. Similarly as in Eq. (23), we can do the sum: The function $s_{(2)n}(t', t'') := s_n(t') s_n(t'') = \text{sinc}[\pi(nl - t')/l] \text{sinc}[\pi(nl - t'')/l]$ is of the bandlimited kind and sampled here at the times nl ; it is reconstructed by the summation including the factor $s_n(t)$, in accordance with the *Sampling Theorem*. However, the bandwidths need to be

considered carefully. By Fourier transformation, one verifies that $s_{(2)n}$ has a doubled bandwidth, $\omega_{max}^{(2)} = 2\pi/l = 2\omega_{max}$, as compared to s_n , which one would guess. This is implemented by writing all appearances of l in terms of $l/2$ and by applying Eq. (12) to yield the sum:

$$\begin{aligned} \sum_{n \in \mathbf{Z}} s_n(t) s_{(2)n}(t', t'') &= \sum_{n \in \mathbf{Z}} \text{sinc}[\pi \frac{nl/2 - t/2}{l/2}] \\ &\quad \cdot \text{sinc}[\pi \frac{nl/2 - t'/2}{l/2}] \text{sinc}[\pi \frac{nl/2 - t''/2}{l/2}] \\ &= \text{sinc}[\pi(t - t')/l] \text{sinc}[\pi(t - t'')/l] . \end{aligned} \quad (28)$$

Thus, we obtain:

$$\psi_{(2)}(t) = l^{-2} \left(\int_{-\infty}^{\infty} dt' \text{sinc}[\pi(t - t')/l] \psi(t') \right)^2 , \quad (29)$$

which expresses $\psi_{(2)}$ in terms of ψ . — In the limit of vanishing discreteness scale, we obtain a simple quadratic term :

$$\lim_{l \rightarrow 0} \psi_{(2)}(t) = (\psi(t))^2 , \quad (30)$$

with the help of Eq. (21). This presents, of course, the expected result. It is *local in time*.

However, we observe that the quadratic term on the right-hand side of Eq. (29) consists in factors which are *nonlocal in time*: the function ψ is integrated over *all* times, weighted by the oscillating and slowly decaying *sinc* kernel. Inserting this result (and corresponding additional terms) into the continuous time version of the discrete analogue of a nonlinear Schrödinger equation, Eq. (24), changes the character of this equation profoundly: it is *not anymore a consistent CA updating rule!* — This should be contrasted with the left-hand side of Eq. (13), which is nonlocal as well. However, this nonlocality is rather mild and refers to two neighbouring instants in such a way that the linear equation can be solved forwards (or backwards) in time step by step, recalling the discussion of initial conditions following Eq. (13) in Sec. 3. With the nonlocality here, due to an anharmonic potential, inserted on the right-hand side of the continuous time equation, this fails.

One may also consider that application of the *Sampling Theorem* to a nonlinear finite difference equation, such as Eq. (24), cannot lead to a consistent continuous dynamics, since the resulting linear and nonlinear terms have different bandwidths (unless an additional cut-off on nonlinear terms is introduced by hand). Which holds for any kind of polynomial nonlinearity.

Thus, we expect nonlocality to be a problem for *any* continuous description based on a form of sampling theory of an underlying discrete CA dynamics, *unless the CA updating rules are linear* in the dynamical variables (as in Sec. 2.). We anticipate this to be the case also if space is discrete, besides time, a situation which can be studied along similar lines [22].

This leaves us with a speculative question: *Could it be that unitary linear evolution in continuous time — which appears to hold universally in QM (leaving aside measurement processes) — is dictated by a local perspective on more general, possibly nonlinear underlying CA dynamics?* In short: *Does locality filter for linearity?*

5. Conclusions

We have briefly reviewed, in Sec. 2., the description of a class of deterministic discrete cellular automata based on an action principle [2, 3, 4, 20]. In particular, we have recalled how this can be mapped with the help of Shannon's sampling theory [18, 19] on a continuous time picture, which resembles in many respects the description of nonrelativistic quantum mechanical objects.

In Sec. 3., we have pointed out the relation between the discrete CA updating rules, which are closely analogous to Hamilton's equations of motion in mechanics, and a modified Schrödinger

equation, which includes additional terms due to the finite discreteness scale l characterizing the CA (and leads to a modified dispersion relation, with energy bounded below and above). This extends to a one-to-one correspondence between the associated conservation laws, between continuous unitary symmetries and their discrete counterparts.

Presently, in Sec. 4., we have paid particular attention to a generalization which incorporates polynomial nonlinearities into the action and equations of motion in a consistent way.

This is motivated by earlier indications that only linear CA equations of motion could be consistent [2, 3] — which appeared a surprising result in view of the essential linearity of QM [14]. It is a fact that too naive implementation of genuine nonlinearities tends to produce overdetermined CA updating rules.

We have found here that introducing a consistent nonlinear generalization of the discrete CA dynamics leads to an inconsistent nonlocality (in time) of the corresponding continuous time description obtained with the help of sampling theory. Furthermore, we have argued that nonlinearity has a detrimental effect (through Poisson brackets based on discrete variational derivatives) on the algebra of observables, as far as it compares with QM [21]. Both would severely spoil any attempt to construct models in the class of Hamiltonian CA which have QM features emerging at large scales (*i.e.* discreteness scale $l \rightarrow 0$) from underlying CA dynamics.

We conclude that nonlinearity must be excluded from the kind of CA model building studied here, in order to maintain locality in the continuous description to the extent possible, when ordinary derivatives are replaced by finite differences beneath.

Yet one may wonder about effects of nonlinear CA processes, if they influence only some dynamical variables. Which could be different ones than those commonly described by a linear and local evolution in quantum theory. Is there room for stochastic phenomena that manifest themselves in QM measurements?

In any case, in order to turn our observations of surprising connections between the properties of cellular automata and the quantum mechanical features of more familiar physical objects into a theory, as proposed by G. 't Hooft [1], several immediate problems call for attention. We should understand how composite systems fare in this context. Which is a prerequisite to analyze the CA analogue of QM measurement processes and, more generally, the role of superposition principle and entanglement. Above all, we need to understand how aspects of relativity and of the physics of spacetime come into play here.

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